Fourier Series Based Nonminimum Phase Model for Second- and Higher-Order Statistical Signal Processing

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Abstract

In the paper, a parametric Fourier series based model (FSBM) for or as an approximation to an arbitrary nonminimum-phase linear time-invariant (LTI) system is proposed for statistical signal processing applications where a model for LTI systems is needed. Based on the FSBM, a (minimum-phase) linear prediction error (LPE) filter for amplitude estimation of the unknown LTI system together with the Cramer Rao (CR) bounds is presented. Then an iterative algorithm for obtaining the optimum mean-square LPE filter with finite data is presented which is also an approximate maximum likelihood algorithm when data are Gaussian. Then three iterative algorithms using higher-order statistics with finite non-Gaussian data are presented for estimating parameters of the FSBM followed by some simulation results to support the efficacy of the proposed algorithms. Finally, we draw some conclusions.

1. Introduction

In many statistical signal processing areas such as signal modeling, power spectral and polyspectral estimation, system identification, deconvolution and equalization, a widely known problem is the identification and estimation of an unknown linear time-invariant (LTI) system $h(n)$ driven by an unknown random signal $u(n)$ with only a given set of output measurements $x(n)$

$$x(n) = u(n) * h(n) = \sum_{k=-\infty}^{\infty} u(k)h(n-k) \qquad (1)$$

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The system function $H(z)$ is often modeled as a parametric rational function such as autoregressive (AR) model, moving average (MA) model and autoregressive moving average (ARMA) model, and therefore finding a rational model approximation to the system from data becomes a parameter estimation problem.

Recently, Chien, Yang and Chi [1] proposed a parametric cumulant based method for estimating the phase of the unknown system $h(n)$ through allpass filtering of measurements $x(n)$ when $x(n)$ is non-Gaussian. Their method is applicable for both 1-D and 2-D systems. They used a Fourier series based model (FSBM) for an allpass filter which leads to a consistent estimate for the system phase by maximizing a single absolute higher-order cumulant of the allpass filter output.

In this paper, an FSBM for or as an approximation to an arbitrary nonminimum-phase LTI system is proposed for applications in the aforementioned statistical signal processing areas. A linear prediction error (LPE) filter based on the FSBM for amplitude estimation of the system is proposed, and then estimation of FSBM (amplitude and phase) parameters is presented followed by some simulations results and conclusions.

2. Nonminimum-phase FSBM

Assume that $h(n)$ is a real nonminimum-phase LTI system with the frequency response $H(\omega) = H(z = \exp\{j\omega\}) = H^*(-\omega)$ defined as

$$H(\omega) = \exp\left\{\sum_{i=1}^{p} \alpha_i \cos(i\omega) + j\sum_{i=1}^{p} \beta_i \sin(i\omega)\right\} \qquad (2)$$

Two advantages of the pth-order nonminimum-phase FSBM defined by (2) over the rational model (i.e., AR, MA and ARMA models) for stable LTI systems are discussed as follows.

The FSBM given by (2) is always a stable IIR system no matter whether it is causal or noncausal. Therefore, in practical applications where the system to be designed is a noncausal stable system such as noncausal inverse filter $1/H(\omega)$ (when $h(n)$ is not minimum-phase) in blind deconvolution and channel equalization, the FSBM given by (2) is more suitable than the ARMA model because the stability issue is never existent for the former, thus leading to more efficient design and processing procedure.

The complex cepstrum $\widetilde{h}(n)$ (inverse Fourier transform of $\ln[H(\omega)]$), which has been used in speech deconvolution, source separation of speech signals and seismic deconvolution, associated with the FSBM given by (2) can be easily shown to be

$$\begin{aligned}\widetilde{h}(n) &= \tfrac{1}{2}\textstyle\sum_{i=1}^{p} \alpha_i\{\delta(n+i)+\delta(n-i)\} \\ &+\tfrac{1}{2}\textstyle\sum_{i=1}^{p} \beta_i\{\delta(n+i)-\delta(n-i)\}\end{aligned} \quad (3)$$

without need of finding poles and zeros of the system [2] as the ARMA model requires.

Next, let us present minimum-phase FSBM and allpass FSBM. The FSBM for $H(\omega)$ given by (2) can also be expressed as

$$H(\omega) = H_{\mathrm{MP}}(\omega) \cdot H_{\mathrm{AP}}(\omega) \quad (4)$$

where the FSBM

$$H_{\mathrm{MP}}(\omega) = \exp\left\{\sum_{i=1}^{p} \alpha_i \cos(i\omega) - j\sum_{i=1}^{p} \alpha_i \sin(i\omega)\right\} \quad (5)$$

can be shown to be a causal stable minimum-phase system with $|H_{\mathrm{MP}}(\omega)| = |H(\omega)|$ and $h_{\mathrm{MP}}(0) = 1$, and $H_{\mathrm{AP}}(\omega)$ is also an allpass FSBM given by

$$H_{\mathrm{AP}}(\omega) = \exp\left\{j\sum_{i=1}^{p} \gamma_i \sin(i\omega)\right\} \quad (6)$$

where

$$\gamma_i = \alpha_i + \beta_i \quad (7)$$

3. FSBM for LPE filters

Let us briefly review the conventional LPE filter for ease of later need for the presentation of the FSBM for LPE filters.

A. Conventional LPE filters

Assume that $x(n)$ is a real stationary random process modeled by (1) where $h(n)$ is a stable LTI system driven by a white noise $u(n)$ with zero mean and variance σ^2. The conventional pth-order LPE filter [3]

$$A_p(z) = 1 + \sum_{i=1}^{p} a_i z^{-i} \quad (8)$$

(a causal FIR filter) processes $x(n)$ such that the prediction error

$$e(n) = x(n) * a_n = x(n) + \sum_{k=1}^{p} a_k x(n-k) \quad (9)$$

has minimum variance or average power $E[e^2(n)]$. The optimum LPE filter $\widehat{A}_p(z)$ is minimum-phase and can be obtained by the following orthogonality principle:

$$E[e(n)x(n-k)] = 0, \quad k = 1,2,...,p \quad (10)$$

leading to a set of symmetric Toeplitz linear equations of $\boldsymbol{a}_p = (a_1, ..., a_p)^T$. When $x(n)$ is an AR(p) Gaussian process, for any unbiased estimates $\widehat{\boldsymbol{a}}_p$ and $\widehat{\sigma^2}$ with finite data, their covariance matrix is lower bounded by the following Cramer Rao (CR) bounds [3]:

$$\mathbf{C}_{\widehat{\boldsymbol{a}}_p, \widehat{\sigma^2}} \geq \frac{\sigma^2}{N}\begin{bmatrix} \mathbf{R}_{xx}^{-1} & \mathbf{0} \\ \mathbf{0}^T & 2\sigma^2 \end{bmatrix} \quad (11)$$

where N is the total number of data and $\mathbf{R}_{xx} = E[\mathbf{x}\mathbf{x}^T]$ in which $\mathbf{x} = (x(n), ..., x(n-p+1))^T$.

B. LPE filters with FSBM

Let the pth-order LPE filter $v_p(n)$ be a causal stable minimum-phase IIR filter with $v_p(0) = 1$ and

$$V_p(\omega) = \exp\left\{\sum_{i=1}^{p} a_i \cos(i\omega) - j\sum_{i=1}^{p} a_i \sin(i\omega)\right\} \quad (12)$$

and thus the prediction error is given by

$$e(n) = x(n) * v_p(n) = x(n) + \sum_{k=1}^{\infty} v_p(k)x(n-k) \quad (13)$$

Note that we have used the same notations a_i for parameters of both the proposed FSBM LPE filter $V_p(\omega)$ and the conventional LPE filter $A_p(z)$. The optimum LPE filter $\widehat{V}_p(\omega)$ is described in the following theorem.

Theorem 1. Assume that $H(\omega)$ is an FSBM given by (2) with order equal to q instead of p, and $e(n)$ is the prediction error given by (13) with the LPE filter order $p \geq q$. Then the optimum LPE filter $\widehat{V}_p(\omega) = 1/H_{\mathrm{MP}}(\omega)$ with $\min\{E[e^2(n)]\} = E[u^2(n)] = \sigma^2$.

The optimum prediction error $e(n)$ must satisfy $\partial E[e^2(n)]/\partial a_k = 0$ from which one can prove the following orthogonality principle:

$$E[e(n)e(n-k)] = r_{ee}(k) = 0, \quad k = 1,2,...,p \quad (14)$$

However, (14) forms a set of nonlinear equations $\boldsymbol{a}_p$. Nevertheless, $e(n)$ will be a white process as p is sufficiently large which implies that $\widehat{V}_p(\omega) = \widehat{A}_p(\omega)$ (identical whitening filter) for $p = \infty$.

Based on Theorem 1, an iterative algorithm is used to estimate or find an approximation to $H_{\mathrm{MP}}(\omega)$ with finite data $x(0)$, $x(1)$, ..., $x(N-1)$ as follows:

Algorithm 1. Amplitude estimation

(S1) Set $p_{\max}$ (maximum of p), parameter L, increment parameter $s \geq 1$, convergence parameter ξ and $t = 0$ (iteration number).

(S2) Set $t = t + 1$, $p = s \times t$ and $\boldsymbol{a}_p = (a_1, ..., a_p)^T$. Search for the minimum of the objective function

$$J(\boldsymbol{a}_p) = \frac{1}{N-L} \sum_{n=L}^{N-1} e^2(n) \tag{15}$$

and the associated optimum $\widehat{\boldsymbol{a}}_p$ by a gradient type iterative optimization algorithm (such as the well-known Fletcher-Powell algorithm) with the initial condition $\boldsymbol{a}_p(0) = (\widehat{\boldsymbol{a}}_{p-s}^T, \mathbf{0}^T)^T$.

(S3) If $p \leq p_{\max}$ and $|J(\widehat{\boldsymbol{a}}_p) - J(\widehat{\boldsymbol{a}}_{p-s})|/J(\widehat{\boldsymbol{a}}_{p-s}) \geq \xi$, go to (S2), otherwise

$$\widehat{H}_{\mathrm{MP}}(\omega) = 1/\widehat{V}_p(\omega) \quad (\text{or } \widehat{\alpha}_i = -\widehat{a}_i) \tag{16}$$

$$\widehat{\sigma^2} = J(\widehat{\boldsymbol{a}}_p) \tag{17}$$

The optimum prediction error $e(n) = x(n) * \widehat{v}_p(n)$ corresponds to amplitude equalized data, and the gradient of $J(\boldsymbol{a}_p)$ with respect to a_k needed in (S2) can be shown to be

$$\frac{\partial J(\boldsymbol{a}_p)}{\partial a_k} = 2\widehat{r}_{ee}(k) = \frac{2}{N-L} \sum_{n=L}^{N-1} e(n)e(n-k) \tag{18}$$

When $H(\omega)$ is a pth-order FSBM and $x(n)$ is Gaussian, it can be shown that both $\widehat{\boldsymbol{\alpha}}_p = -\widehat{\boldsymbol{a}}_p$ and $\widehat{\sigma^2}$ are approximate maximum-likelihood estimates. Moreover, the CR bounds associated with any unbiased estimates $\widehat{\boldsymbol{\alpha}}_p$ and $\widehat{\sigma^2}$ are given by

$$\mathbf{C}_{\widehat{\boldsymbol{\alpha}}_p, \widehat{\sigma^2}} \geq \frac{1}{N} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^T & 2\sigma^4 \end{bmatrix} \tag{19}$$

where $\mathbf{I}$ is a $p \times p$ identity matrix. Note that the CR bounds associated with AR parameters (see (11)) depend on correlations of $x(n)$, while those associated with α_i are uniform and independent of correlations of $x(n)$. The CR bound associated with σ^2 is the same for both FSBM and AR model.

4. Estimation of FSBM parameters

In this section, further with the assumption that $u(n)$ is non-Gaussian with nonzero Mth-order (≥ 3) cumulant (and thus $x(n)$ is also non-Gaussian), three iterative algorithms are to be presented for the estimation of parameters of the FSBM $H(\omega)$ given by (2).

The first two algorithms estimate the system amplitude using Algorithm 1 and system phase using Chien, Yang and Chi's phase estimation algorithm [1] which maximizes a single absolute Mth-order (≥ 3) sample cumulant, denoted $|\widehat{C}_{M,y}|$, of the phase equalized (allpass filtered) data

$$y(n) = x(n) * g_{\mathrm{AP}}(n) \tag{20}$$

where $g_{\mathrm{AP}}(n)$ is a pth-order allpass FSBM

$$G_{\mathrm{AP}}(\omega) = \exp\left\{ j \sum_{i=1}^{p} b_i \sin(i\omega) \right\} \tag{21}$$

It has been shown in [1] that the optimum $\widehat{G}_{\mathrm{AP}}(\omega)$ turns out to be a phase equalizer, i.e.,

$$\arg\{\widehat{G}_{\mathrm{AP}}(\omega)\} = -\arg\{H(\omega)\} + \omega\tau \tag{22}$$

Because $|\widehat{C}_{M,y}|$ is a highly nonlinear function of b_i, one can use gradient type iterative algorithms for finding the optimum b_i. It has also been proven in [1] that

$$\frac{\partial y(n)}{\partial b_i} = \frac{1}{2}\{y(n+i) - y(n-i)\} \tag{23}$$

which is needed for computing the gradient of $|\widehat{C}_{M,y}|$ with respect to b_i. Next, let us present Algorithms 2 and 3, respectively,

Algorithm 2.

(S1) Estimate $H_{\mathrm{MP}}(\omega)$ and σ^2 using Algorithm 1.

(S2) Find the optimum allpass FSBM $G_{\mathrm{AP}}(\omega)$ given by (21) using a gradient type iterative algorithm such that $|\widehat{C}_{M,y}|$ is maximum where $y(n) = x(n) * g_{\mathrm{AP}}(n)$. Then obtain the estimate $\widehat{\beta}_i = -\widehat{b}_i$.

Algorithm 3.

(S1) Estimate $H_{\mathrm{MP}}(\omega)$ and σ^2, and obtain the optimum prediction error $e(n) \simeq u(n) * h_{\mathrm{AP}}(n)$ using Algorithm 1.

(S2) Find the optimum allpass FSBM $G_{\mathrm{AP}}(\omega)$ given by (21) using a gradient type iterative algorithm such that $|\widehat{C}_{M,y}|$ is maximum where $y(n) = e(n) * g_{\mathrm{AP}}(n)$. Then obtain the estimate $\widehat{\gamma}_i = -\widehat{b}_i$.

The last algorithm (Algorithm 4) estimates the system amplitude and phase simultaneously using inverse filter criteria [4-7]. Chi and Wu [4] proposed a family of inverse filter criteria which includes Tugnait's criteria [5], Wiggins' criterion [6] and Shalvi and Weinstein's criterion [7] as special cases. The inverse filter $h_{\text{INV}}(n)$ is estimated by maximizing

$$J_{r,m} = \frac{|\widehat{C}_{m,\widehat{u}}|^r}{|\widehat{C}_{r,\widehat{u}}|^m} \tag{24}$$

where r is even and $m > r$, and

$$\widehat{u}(n) = x(n) * h_{\text{INV}}(n) \tag{25}$$

It has been shown in [4] that $\widehat{u}(n) = bu(n - \tau)$ when $h(n)$ is an arbitrary stable LTI system where b is a scale factor and τ is an unknown time delay. Next, let us present Algorithm 4.

Algorithm 4.

(S1) Set integer $r \geq 2$ (even) and integer $m > r$. Let $H_{\text{INV}}(\omega) = 1/H(\omega)$ as given by (2) with α_i and β_i replaced by $-\alpha_i$ and $-\beta_i$, respectively.

(S2) Find the optimum $H_{\text{INV}}(\omega)$ (i.e., α_i and β_i) using a gradient type iterative algorithm such that $J_{r,m}$ is maximum. Then σ^2 is estimated as the sample variance of the obtained optimum inverse filter output $\widehat{u}(n)$.

It can be shown that

$$\frac{\partial \widehat{u}(n)}{\partial \alpha_i} = -\frac{1}{2}\{\widehat{u}(n+i) + \widehat{u}(n-i)\} \tag{26}$$

$$\frac{\partial \widehat{u}(n)}{\partial \beta_i} = -\frac{1}{2}\{\widehat{u}(n+i) - \widehat{u}(n-i)\} \tag{27}$$

which are needed for computing the gradient of $J_{r,m}$ with respect to α_i and β_i in (S2), respectively.

Notice that when $H(\omega)$ is not minimum-phase, the FSBM (noncausal stable IIR system) is well suited for the noncausal inverse filter $H_{\text{INV}}(\omega) = 1/H(\omega)$ as mentioned in Section 2. When $H(\omega)$ is an FSBM, the optimum $\widehat{u}(n) \simeq u(n)$ (without scale factor and time delay between $\widehat{u}(n)$ and $u(n)$).

5. Simulation results

In this section, let us show two sets of simulation results. For the first simulation, the driving input $u(n)$ was assumed to be a zero-mean i.i.d. Exponentially distributed random sequence. An FSBM of order $p = 5$ was used for the system $H(\omega)$, whose amplitude and phase (solid lines) are shown in Figures 1a and 1b, respectively. Thirty independent runs were performed using Algorithm 3 with $p_{\max} = s = 5$, $L = 0$ and $M = 3$. Mean (dashed line) and mean±standard deviation (dotted lines) of the obtained thirty amplitude and phase estimates for $N = 1024$ and $SNR = 20$ dB (white Gaussian noise) are also shown in Figures 1a and 1b, respectively. One can see that both the amplitude and phase estimates are unbiased with small variance. Moreover, the results obtained by Algorithms 2 and 4 are similar to those shown in Figure 1.

The second set of simulation results for seismic deconvolution was obtained with $u(n)$ assumed to be a Bernoulli-Gaussian sequence and the system (source wavelet) $h(n)$ to be a third-order nonminimum-phase causal ARMA system (taken from [4]) instead of an FSBM. Figure 2a shows the synthetic data $x(n)$ for $N = 512$ and $SNR = 20$ dB (white Gaussian noise). Figures 2b and 2c show the (noncausal) estimate $\widehat{h}(n)$ (dotted line) and the deconvolved signal $\widehat{u}(n)$ (dotted line), respectively, obtained using Algorithm 4 with $p = 12$, $r = 2$ and $m = 4$, where the scale factor as well as the time delay between $\widehat{h}(n)$ and $h(n)$ (solid line) and the time delay between $\widehat{u}(n)$ and $u(n)$ (solid line) were artificially removed. One can see that $\widehat{h}(n)$ and $\widehat{u}(n)$ are good approximations to $h(n)$ and $u(n)$, respectively. The above simulation results support the efficacy of the proposed algorithms.

6. Conclusions

We have presented an FSBM for or as an approximation to an arbitrary nonminimum-phase LTI system for applications in the statistical signal processing areas mentioned in the introduction section. Based on the FSBM, an LPE filter for amplitude estimation together with the CR bounds, an algorithm for obtaining the optimum LPE filter, and three algorithms for estimating FSBM parameters were presented. All the gradient type iterative optimization algorithms used in the proposed algorithms have a computationally efficient parallel structure (FIR filter banks with only two nonzero coefficients 1/2 or −1/2) (see (23), (26) and (27)). However, the gradient computation associated with Algorithm 1 (see (18)) does not need any further processing to the prediction error $e(n)$. Finally, two sets of simulation results were presented to support the efficacy of the proposed algorithms.

References

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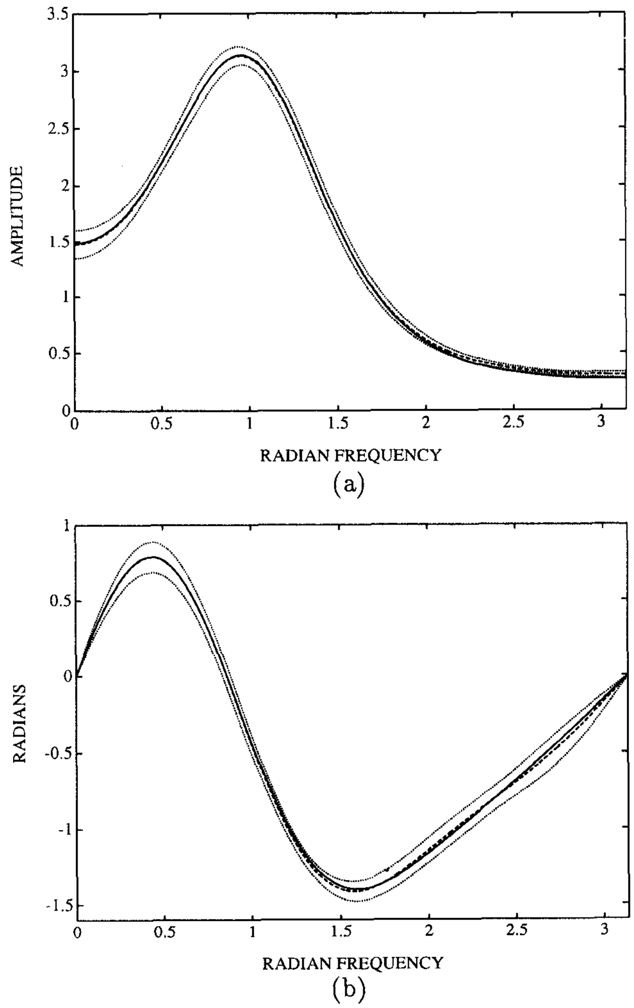

Figure 1. Simulation results for $N = 1024$ and $SNR = 20$ dB using Algorithm 3. Mean (dashed lines) and mean$\pm$standard deviation (dotted lines) of thirty (a) amplitude and (b) phase estimates together with the amplitude and phase (solid lines) of the system.

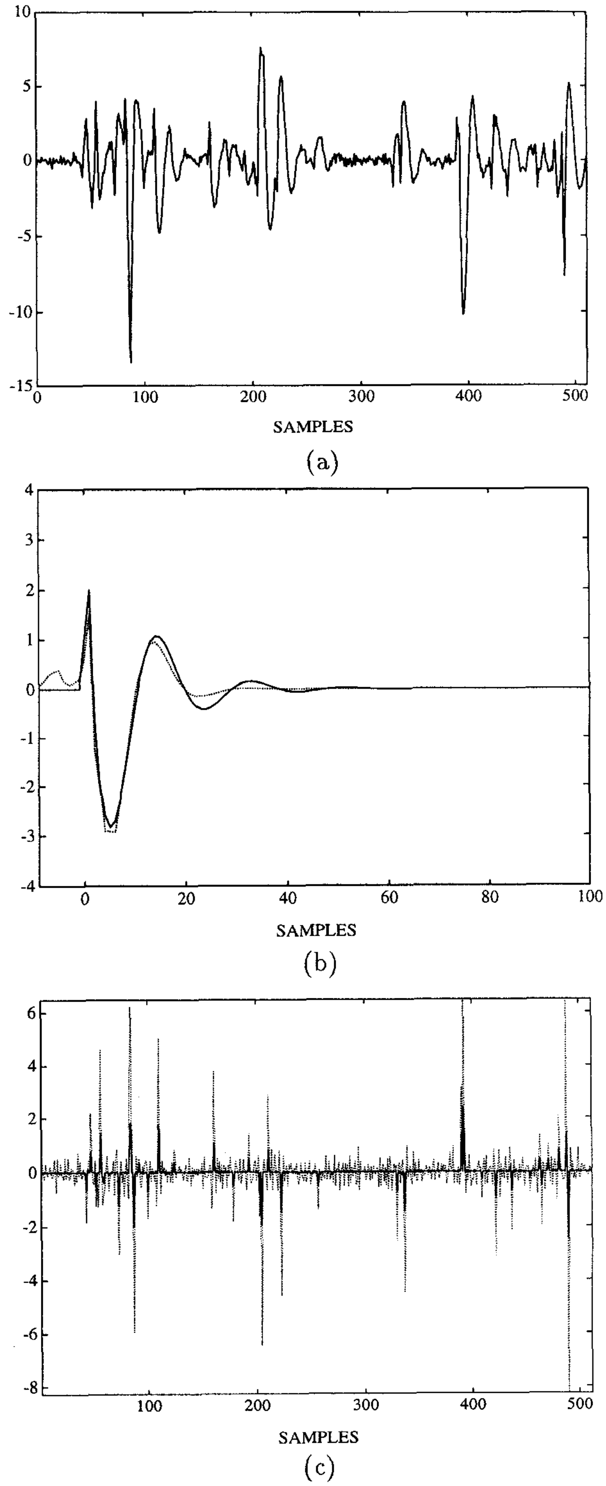

Figure 2. Simulation results for seismic deconvolution ($N = 512$ and $SNR = 20$ dB) using Algorithm 4. (a) Synthetic data $x(n)$; (b) source wavelet $h(n)$ (solid line) and estimate $\widehat{h}(n)$ (dotted line); and (c) input $u(n)$ (solid line) and deconvolved signal $\widehat{u}(n)$ (dotted line).